

THE CONFIGURATION BASIS OF A LIE ALGEBRA AND ITS DUAL

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ABSTRACT. We use the Lie coalgebra and configuration pairing framework presented previously in [11] to derive a new, left-normed monomial basis for free Lie algebras (built from associative Lyndon-Shirshov words), as well as a dual monomial basis for Lie coalgebras. Our focus is on computational dexterity gained by using the configuration framework and basis. We include several explicit examples using the dual coalgebra basis and configuration pairing to perform Lie algebra computations. As a corollary of our work, we get a new multiplicative basis for the shuffle algebra.

1. INTRODUCTION

Let V be a finite dimensional vector space with ordered basis $v_1 < \dots < v_d$. The (associative) Lyndon-Shirshov words on the alphabet v_1, \dots, v_d are finite words which are lexicographically minimal among their cyclic permutations. Various different methods (see e.g. [2], [1]) are present in the literature, constructing different bases for the free Lie algebra on V starting from Lyndon-Shirshov words. We will derive a new basis for LV from Lyndon-Shirshov words, as well as a dual basis of monomials in the graph presentation of the cofree conilpotent Lie coalgebra EV^* on V^* using the configuration pairing and graph coalgebras framework of [11].

Since our basis is natural from the point of view of the configuration pairing, we call it the configuration basis of a Lie algebra. We emphasize that the surprising property of the configuration basis is that it has a dual Lie coalgebra basis of graph monomials. While some previous work has considered duality (e.g. [3]) in the formation of bases for Lie algebras, previously duality was formal, rather than using explicit presentations, and work occurred in the universal enveloping algebra of Lie algebras, rather than on the Lie algebras themselves.

Note that even though our starting point is the set of Lyndon words, the configuration basis is not a Hall basis. However, it still satisfies some of the same nice properties as Hall bases. For example, writing Lie bracket expressions in terms of the configuration basis yields integer coefficients (Corollary 4.4) and the Lie polynomial of the configuration basis can be written as shuffles of higher ordered elements (Theorem 6.4). The latter property implies that the configuration basis may be used for Gröbner basis calculations.

The paper is structured as follows. In Section 2 we recall the Lie coalgebras framework introduced in [10] and [11]. We organize ideas slightly differently than [11] and give extra computational examples to clarify the framework and its use in this setting. The central idea needed is the configuration pairing of Lie algebras and coalgebras. This will be described by writing Lie algebra elements as trees and Lie coalgebra elements as graphs.

In Section 3 we introduce a new grading on the set of Lyndon-Shirshov words \mathcal{B} [1], [7]. A simple word is $x \dots x y_1 \dots y_\ell$ with $x \neq y_i$. Words are graded by classifying them as simple words, simple words of simple words, simple words of those, etc.

In Section 4 we recursively define bracketing and graph maps \mathcal{L} and \mathcal{G} making Lie algebra and coalgebra elements from Lyndon-Shirshov words. Our main theorem shows that \mathcal{LB} and \mathcal{GB} are dual monomial bases

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for the free Lie algebra and cofree conilpotent Lie coalgebra via the configuration pairing. The set \mathcal{LB} is the configuration basis of the free Lie algebra LV .

In Sections 5 and 6 we give examples using the configuration pairing to write Lie expressions in terms of the configuration basis. We also recall the classical Harrison model of conilpotent Lie coalgebras as words modulo shuffles and note that in this model the Lyndon words themselves form a basis. In examples, we show how the configuration pairing, which is not part of the classical picture, can be used to perform Lie algebra calculations using the Lyndon word basis for the Harrison model of EV^* .

In Sections 7 and 9 we compare the configuration basis to some other bases present in the literature, connecting with classical Lie models, and outline some possible avenues of future interest. In Section 8 we also construct an alternate set of words $\hat{\mathcal{B}}$ which can be used as a replacement for the Lyndon-Shirshov words. The set $\hat{\mathcal{B}}$ is also a new multiplicative basis for the shuffle algebra.

2. GRAPHS, LIE COALGEBRAS, AND THE CONFIGURATION PAIRING

We begin with a summary of pertinent results from [11] and [10] slightly rephrased and specialized for the convenience of our setting. Throughout this section we will say tree and graph to mean rooted, binary tree embedded in the upper-half plane and oriented, connected, acyclic graph. Trees will be denoted T and graphs, G . Labeled trees and graphs, $\tau = (T, l_T)$ and $\gamma = (G, l_G)$ are trees and graphs with maps $l_T : \text{Leaves}(T) \rightarrow S_T$ and $l_G : \text{Vertices}(G) \rightarrow S_G$ where S_T, S_G are labeling sets.

Our shorthand for writing labeled trees and graphs will be to write corresponding labels in place of their leaves and vertices. We do not require l_T or l_G to be injective or surjective.

Let V be a finite dimensional vector space with dual V^* . Write $\text{Tr}(V)$ and $\text{Gr}(V^*)$ for the vector spaces generated by trees and graphs with labels from V and V^* , modulo multilinearity in the labels. The vector space $\text{Tr}(V)$ has a standard product defined on monomials as $\tau_1 \otimes \tau_2 \mapsto [\tau_1, \tau_2] = \tau_1 \frown \tau_2$. The vector space $\text{Gr}(V^*)$ has an anti-commutative coproduct defined on monomials as $\gamma \mapsto]\gamma[= \sum_e \gamma_1^e \otimes \gamma_2^e - \gamma_2^e \otimes \gamma_1^e$ where \sum_e is a sum over all edges of G and γ_1^e, γ_2^e are the graphs obtained from γ by removing the edge e which points to the subgraph γ_2^e .

It is a standard fact that LV , the free Lie algebra on V , is isomorphic as algebras to $\text{Tr}(V)$ modulo the locally defined anti-symmetry and Jacobi relations:

$$\begin{aligned} \text{(anti-symmetry)} \quad & \begin{array}{c} T_1 \quad T_2 \\ \diagdown \quad \diagup \\ R \end{array} = - \begin{array}{c} T_2 \quad T_1 \\ \diagdown \quad \diagup \\ R \end{array} \\ \text{(Jacobi)} \quad & \begin{array}{c} T_1 \quad T_2 \quad T_3 \\ \diagdown \quad \diagup \quad \diagup \\ R \end{array} + \begin{array}{c} T_2 \quad T_3 \quad T_1 \\ \diagdown \quad \diagup \quad \diagup \\ R \end{array} + \begin{array}{c} T_3 \quad T_1 \quad T_2 \\ \diagdown \quad \diagup \quad \diagup \\ R \end{array} = 0, \end{aligned}$$

where R, T_1, T_2, T_3 stand for arbitrary (possibly trivial subtrees) which are not modified in these operations. ($\text{Tr}(V)$ itself is isomorphic to the free nonassociative binary algebra on V , aka the free magma on V .) From [11] the cofree conilpotent Lie coalgebra on V^* , written EV^* , is isomorphic as coalgebras to $\text{Gr}(V^*)$ modulo the locally defined arrow-reversing and Arnold relations:

$$\begin{aligned} \text{(arrow-reversing)} \quad & \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ a \end{array} = - \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ \diagdown \quad \diagup \quad \diagdown \\ a \end{array} \\ \text{(Arnold)} \quad & \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ a \end{array} + \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ \diagdown \quad \diagup \quad \diagup \\ a \end{array} + \begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ \diagup \quad \diagdown \quad \diagup \\ a \end{array} = 0, \end{aligned}$$

where a, b , and c stand for vertices in a graph which is fixed outside of the indicated area. ($\text{Gr}(V^*)$ itself is closely related to the cofree preLie coalgebra on V^* .)

Remark 2.1. In [11, Prop. 3.2] work is restricted to graded, 1-reduced vector spaces. This requirement is needed to have EV be the cofree Lie coalgebra. Removing graded, 1-reduced results in EV being the cofree,

conilpotent Lie coalgebra. This follows from [11, Prop. 3.14], which is independent of [11, Prop. 3.2]. The difference between cofree Lie coalgebras and cofree, conilpotent Lie coalgebras is the presence of infinite graphs, which are not needed in the present application.

The core of the $EV \cong \text{Gr}(V)/\sim$ proof in [11], and backbone of the current paper, is the configuration pairing of graphs and trees, introduced in [10] and extended to $\text{Gr}(V^*)$ and $\text{Tr}(V)$ in [11]. Given an isomorphism $\sigma : \text{Vertices}(G) \rightarrow \text{Leaves}(T)$, define $\beta_\sigma : \text{Edges}(G) \rightarrow \{\text{internal vertices of } T\}$ by sending the edge $a \nearrow b$ to the internal vertex closest to the root of T on the path from leaf $\sigma(a)$ to leaf $\sigma(b)$. For unlabeled graphs and trees, the σ -configuration pairing of G and T is

$$\langle G, T \rangle_\sigma = \begin{cases} \prod_{e \in E(G)} \text{sgn}(\beta_\sigma(e)) & \text{if } \beta_\sigma \text{ is surjective,} \\ 0 & \text{otherwise} \end{cases}$$

where \prod_e is a product over all edges of G , and $\text{sgn}(\beta_\sigma(a \nearrow b)) = \pm 1$ depending on whether leaf $\sigma(a)$ is left or right of leaf $\sigma(b)$ in the planar embedding of T .

Example 2.2. Following is the map β_σ for two different isomorphisms σ of the vertices and leaves of a fixed graph and tree. The different isomorphisms are indicated by the numbering of the vertices and leaves.



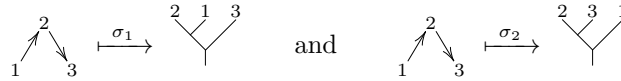
In the first example, $\text{sgn}(\beta_{\sigma_1}(e_1)) = -1$ and $\text{sgn}(\beta_{\sigma_1}(e_2)) = 1$. In the second example, $\text{sgn}(\beta_{\sigma_2}(e_1)) = 1$ and $\text{sgn}(\beta_{\sigma_2}(e_2)) = -1$. The associated σ -configuration pairings are $\langle G, T \rangle_{\sigma_1} = -1$ and $\langle G, T \rangle_{\sigma_2} = 0$.

Definition 2.3. On monomials $\gamma = (G, l_G) \in \text{Gr}(V^*)$ and $\tau = (T, l_T) \in \text{Tr}(V)$ let

$$\langle \gamma, \tau \rangle = \sum_{\sigma: V(G) \xrightarrow{\cong} L(T)} \left(\langle G, T \rangle_\sigma \prod_{v \in V(G)} \langle l_G(v), l_T(\sigma(v)) \rangle \right)$$

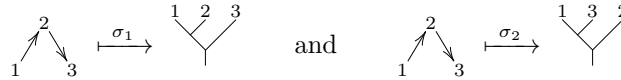
where \sum_σ is a sum over all isomorphisms $\sigma : \text{Vertices}(G) \rightarrow \text{Leaves}(T)$ and \prod_v is a product over all vertices of G . If there are no isomorphisms σ , then $\langle \gamma, \tau \rangle = 0$. The configuration pairing is \langle, \rangle extended to $\text{Gr}(V^*) \times \text{Tr}(V)$ by multilinearity.

Example 2.4. The configuration pairing $\left\langle \begin{array}{c} a^* \\ b^* \nearrow b^* \end{array}, \begin{array}{c} a \quad b \quad b \\ \diagdown \quad | \quad \diagup \end{array} \right\rangle = -2$. The isomorphisms



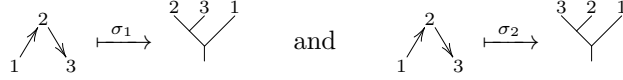
between $\text{Vertices}(G)$ and $\text{Leaves}(T)$ are the only two which will give $\prod_v \langle l_G(v), l_T(\sigma(v)) \rangle \neq 0$. These pair $\langle G, T \rangle_{\sigma_1} = -1$ and $\langle G, T \rangle_{\sigma_2} = -1$.

Example 2.5. The configuration pairing $\left\langle \begin{array}{c} b^* \\ a^* \nearrow b^* \end{array}, \begin{array}{c} a \quad b \quad b \\ \diagdown \quad | \quad \diagup \end{array} \right\rangle = 1$. The isomorphisms



between $\text{Vertices}(G)$ and $\text{Leaves}(T)$ are the only two which will give $\prod_v \langle l_G(v), l_T(\sigma(v)) \rangle \neq 0$. These pair $\langle G, T \rangle_{\sigma_1} = 1$ and $\langle G, T \rangle_{\sigma_2} = 0$.

Example 2.6. The configuration pairing $\left\langle \begin{array}{c} b^* \\ \swarrow \searrow \\ a^* \quad b^* \end{array}, \begin{array}{c} b \quad b \quad a \\ \swarrow \downarrow \searrow \end{array} \right\rangle = 0$. The isomorphisms



between $\text{Vertices}(G)$ and $\text{Leaves}(T)$ are the only two which will give $\prod_v \langle l_G(v), l_T(\sigma(v)) \rangle \neq 0$. These pair $\langle G, T \rangle_{\sigma_1} = -1$ and $\langle G, T \rangle_{\sigma_2} = 1$.

From [11], the configuration pairing vanishes on the ideal of $\text{Tr}(V)$ generated by the anti-symmetry and Jacobi identities and on the coideal of $\text{Gr}(V)$ generated by the arrow-reversing and Arnold identities. Thus the configuration pairing descends to a pairing between $\text{E}(V^*)$ and $\text{L}(V)$. Furthermore we have the following.

Theorem 2.7 (3.11 of [11]). *Let V be a finite dimensional vector space over a field of characteristic zero. The configuration pairing of LV and EV^* is a perfect pairing.*

Theorem 2.8 (3.14 of [11]). *Given $\gamma \in \text{EV}^*$ and $[\tau_1, \tau_2] \in \text{LV}$,*

$$\langle \gamma, [\tau_1, \tau_2] \rangle = \sum_i \langle \alpha_i, \tau_1 \rangle \langle \beta_i, \tau_2 \rangle$$

where $[\gamma] = \sum_i \alpha_i \otimes \beta_i$.

Remark 2.9. One corollary of the current work is that we may remove the “characteristic zero” assumption from Theorem 2.7 using a dimension argument.

For more detail on the foundations and interpretations of the configuration pairing, see [13].

3. SIMPLE WORDS

Given a finite dimensional vector space V , choose an ordered basis $v_1 < \dots < v_d$ and write \mathcal{A} for the set of all finite words written using the alphabet $\{v_1, \dots, v_d\}$. Let \mathcal{B} be the set of (associative) Lyndon-Shirshov words in \mathcal{A} . Explicitly, the ordering of its alphabet induces a lexicographical ordering on the words \mathcal{A} . Let \mathcal{B} be the collection of words $\omega \in \mathcal{A}$ where ω has smaller ordering than any of its cyclic permutations.

Example 3.1. For simplicity, write 1 for v_1 , 2 for v_2 , etc.

- 112 $\in \mathcal{B}$ but not 121 or 211.
- 111122, 111212 $\in \mathcal{B}$ but not 112112.

The following proposition is classical, and is proven by a simple counting argument.

Proposition 3.2. *\mathcal{B} satisfies the Witt formula:*

$$\#\{\text{elements of length } n\} = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m}$$

where μ is the Möbius function.

Definition 3.3. A simple word in an ordered alphabet is $\overbrace{x \dots x}^k y_1 \dots y_\ell = x^k y_1 \dots y_\ell$ where $x \neq y_i$ for all i (and $k, \ell > 0$). Two simple words are compatible if their initial letters are the same. Compatible simple words are ordered lexicographically via the ordering of their alphabet.

Let $\mathcal{A}_0 \subset \mathcal{A}$ be the subset of singleton words and $\mathcal{A}_1 \subset \mathcal{A}$ be the subset of simple words. Recursively define $\mathcal{A}_m \subset \mathcal{A}$ for $m > 1$ as the simple words in an ordered alphabet of compatible words in \mathcal{A}_{m-1} . Note that to be in \mathcal{A}_m , a word must be length at least 2^m .

Example 3.4. For clarity, in the following we will insert parenthesis in the \mathcal{A}_2 , \mathcal{A}_3 , and \mathcal{A}_4 examples to indicate subwords from lower levels. Suppose the ordered alphabet \mathcal{A}_0 is $\{1 < 2 < \dots < 9\}$.

- $112 < 12 < 122 < 13 \in \mathcal{A}_1$.
- $(112)(112)(122) < (112)(12) \in \mathcal{A}_2$.
- $((112)(112)(12))((112)(1122)) < ((112)(112)(12))((112)(12))((112)(12)) \in \mathcal{A}_3$.
- $((112)(112)(12))((112)(12))((112)(122)) \Big(((112)(112)(12))((112)(122)) \Big) \in \mathcal{A}_4$.

It is quick to check that the \mathcal{A}_m are disjoint and $\mathcal{A} = \cup_m \mathcal{A}_m$. The key fact is that decomposition of a word into the form $\omega^k \psi_1 \dots \psi_\ell$ with $\omega, \psi_i \in \mathcal{A}_{m-1}$ compatible is unique. When we write $\omega^k \psi_1 \dots \psi_\ell \in \mathcal{A}_m$ our implication will be that the presented decomposition is the unique decomposition into compatible words of \mathcal{A}_{m-1} . For each m define $\mathcal{B}_m = \mathcal{B} \cap \mathcal{A}_m$. The examples from 3.4 are all members of \mathcal{B} . The ordering of compatible words is chosen so that the following is true.

Lemma 3.5. $\omega^k \psi_1 \dots \psi_\ell \in \mathcal{B}_m$ if and only if $\omega, \psi_i \in \mathcal{B}_{m-1}$ and $\omega < \psi_i$ for all i (in the ordering of compatible words of \mathcal{A}_{m-1}).

Proof Sketch. It is enough to show that the ordering on compatible words of \mathcal{A}_{m-1} (as words in an alphabet of compatible words of \mathcal{A}_{m-2}) coincides with their lexicographical ordering as words of \mathcal{A} . Induct. \square

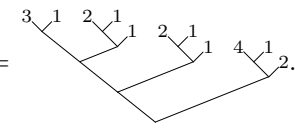
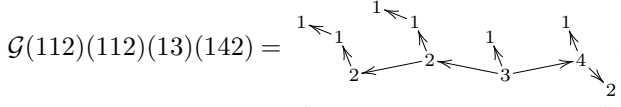
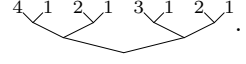
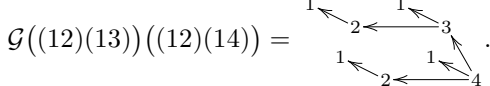
4. THE CONFIGURATION BASIS

Define maps $\mathcal{L} : \mathcal{B} \rightarrow \text{Tr}(V)$ and $\mathcal{G} : \mathcal{B} \rightarrow \text{Gr}(V^*)$ recursively as follows. On \mathcal{B}_0 , $\mathcal{L}v_i = v_i$ and $\mathcal{G}v_i = v_i^*$ (the graph with a single vertex labeled by v_i^*). Call the single vertex of $\mathcal{G}v_i$ the pivot vertex. For $\omega^k \psi_1 \dots \psi_\ell \in \mathcal{B}_m$ define

$$\begin{aligned} \mathcal{L} : \omega^k \psi_1 \dots \psi_\ell &\mapsto [\overbrace{[[[[[\mathcal{L}\psi_1, \mathcal{L}\omega], \dots], \mathcal{L}\omega], \mathcal{L}\psi_2], \dots], \mathcal{L}\psi_\ell}] \\ \mathcal{G} : \omega^k \psi_1 \dots \psi_\ell &\mapsto \begin{array}{c} \mathcal{G}\omega \quad \mathcal{G}\omega \quad \mathcal{G}\psi_2 \quad \mathcal{G}\psi_\ell \\ \nwarrow \quad \swarrow \quad \nwarrow \quad \swarrow \\ \dots \quad \mathcal{G}\psi_1 \quad \dots \end{array} \end{aligned}$$

where the subgraph $\mathcal{G}\omega$ appears k times and the arrows connecting the subgraphs above connect their pivot vertices. The pivot vertex of the above graph is inherited from $\mathcal{G}\psi_1$.

Example 4.1. Following are some examples of $\mathcal{L}\omega$ and $\mathcal{G}\omega$ for $\omega \in \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$. For clarity we will draw the larger bracket expressions also as trees when writing \mathcal{L} and we will neglect $*$ when writing \mathcal{G} .

- $\mathcal{L}12 = [2, 1]$ and $\mathcal{L}112 = [[2, 1], 1]$ and $\mathcal{L}11234 = \left[\left[\left[[2, 1], 1 \right], 3 \right], 4 \right]$.
 $\mathcal{G}12 = \begin{array}{c} 1 \\ \swarrow \\ 2 \end{array}$ and $\mathcal{G}112 = \begin{array}{c} 1 \\ \swarrow \quad \nwarrow \\ 1 \quad 2 \end{array}$ and $\mathcal{G}11234 = \begin{array}{c} 1 \\ \swarrow \quad \nwarrow \quad \swarrow \quad \nwarrow \\ 1 \quad 2 \quad 3 \quad 4 \end{array}$.
- $\mathcal{L}(112)(112)(13)(142) = \left[\left[\left[[3, 1], [2, 1], 1 \right], [2, 1], 1 \right], [4, 1], 2 \right] =$ 
 $\mathcal{G}(112)(112)(13)(142) =$ 
- $\mathcal{L}((12)(13))((12)(14)) = \left[[4, 1], [2, 1], [3, 1], [2, 1] \right] =$ 
 $\mathcal{G}((12)(13))((12)(14)) =$ 

Write \mathcal{GB} and \mathcal{LB} for the images of the set of Lyndon words under \mathcal{G} and \mathcal{L} .

Theorem 4.2. *Bracket expressions \mathcal{LB} and graph expressions \mathcal{GB} are dual vector space bases for the free Lie algebra LV and the cofree conilpotent Lie coalgebra EV^* .*

Proof. Applying Proposition 3.2, it is enough to show that \mathcal{GB} and \mathcal{LB} pair perfectly to prove both that the sets \mathcal{GB} and \mathcal{LB} are each independent, and thus bases, and also that they are dual bases.

Let $v, \omega \in \mathcal{B}$, and suppose $\langle \mathcal{G}\omega, \mathcal{L}v \rangle \neq 0$. Fix a bijection $\sigma : \text{Vertices}(\mathcal{G}\omega) \xrightarrow{\cong} \text{Leaves}(\mathcal{L}v)$ so that its term in the configuration pairing is nonzero. Note that σ induces a bijection between the letters of v and those of ω with repetition. Let x be the minimal letter in v and ω . This letter is also their initial letter, since they are words in \mathcal{B} .

From the definition of \mathcal{L} , the innermost brackets of $\mathcal{L}v$ are of the form $[y_{i,1}, x]$ for some letters $y_{i,1}$. There must be corresponding edges in $\mathcal{G}\omega$ between vertices labeled x^* and $y_{i,1}^*$ corresponding under σ .

From the definition of \mathcal{G} , these edges must be $x^* \xleftarrow{y_{i,1}^*}$.

Further, $\mathcal{L}v$ breaks into a series of maximal length bracket expressions $[[[y_{i,1}, x], \dots x], y_{i,2}], \dots y_{i,\ell_i}]$ with $x \neq y_{i,j} \in \mathcal{A}_0$. A nonzero pairing with $[[y_{i,1}, x], x]$ implies that there is an edge from the graph $x^* \xleftarrow{y_{i,1}^*}$ to a vertex labeled x^* in $\mathcal{G}\omega$. Since non- x^* vertices are connected to at most one x^* in graphs $\mathcal{G}\omega$, this edge must be $x^* \xleftarrow{y_{i,1}^*}$. Continuing in this manner, the bracket expression $[[y_{i,1}, x], \dots x]$ must correspond to a subgraph $x^* \xleftarrow{y_{i,1}^*} \dots \xleftarrow{y_{i,1}^*} x^*$.

The next bracket, with $y_{i,2}$, implies that there must be an edge from this subgraph to a vertex labeled $y_{i,2}^*$ of $\mathcal{G}\omega$. From the structure of \mathcal{G} , the edge cannot connect to an x^* , so it must connect to $y_{i,1}^*$. The next bracket, with $y_{i,3}$, implies that there is an edge from this subgraph to the corresponding vertex $y_{i,3}^*$ in $\mathcal{G}\omega$. If the edge came from $y_{i,1}^*$, then the structure of $\mathcal{G}\omega$ implies that one of the vertices $y_{i,2}^*$ and $y_{i,3}^*$ would also have an edge to a vertex x^* . This is not possible, because this vertex would correspond to an x in some other maximal length bracket expression $[[[y_{j,1}, x], \dots x], y_{j,2}], \dots, y_{j,\ell_j}]$ of $\mathcal{L}v$, implying the presence of a subgraph $y_{i,k}^* \xleftarrow{x^*} \dots \xleftarrow{x^*} y_{j,1}^*$ in $\mathcal{G}\omega$.

The next bracket, with $y_{i,4}$, implies an edge in $\mathcal{G}\omega$ from this subgraph to a corresponding $y_{i,4}^*$. The edge cannot connect to x^* or $y_{i,1}^*$ for the reasons already stated and it cannot connect to $y_{i,2}^*$ because non-pivot vertices in $\mathcal{G}\omega$ are at most bivalent. Continuing in this way, we get that each maximal length bracket expression in $\mathcal{L}v$ must correspond to a subgraph $x^* \xleftarrow{y_{i,1}^*} \dots \xleftarrow{y_{i,1}^*} y_{i,2}^* \xrightarrow{y_{i,1}^*} \dots \xrightarrow{y_{i,1}^*} y_{i,\ell_i}^*$ of $\mathcal{G}\omega$. Both the bracket expression and the subgraph expression above correspond to an \mathcal{A}_1 subword $x^k y_{i,1} \dots y_{i,\ell_i}$ of v and ω .

Thus σ gives a bijection of \mathcal{A}_1 subwords of ω and v . Let ξ be the minimal \mathcal{A}_1 subword of v and ω . By Lemma 3.5, this \mathcal{A}_1 subword is also their initial \mathcal{A}_1 subword. Continue by induction (take the \mathcal{A}_1 subwords as the alphabet and look at bracket expressions and graphs of them, etc). Since v is finite, it is in \mathcal{B}_m for some finite m . At this stage, we will have identified $\omega = v$.

It remains only to show that $\langle \mathcal{G}\omega, \mathcal{L}\omega \rangle = 1$. But this is clear. Recursively apply the calculation $\langle \mathcal{G}x^k y_1 \dots y_\ell, \mathcal{L}x^k y_1 \dots y_\ell \rangle = 1$ using the fact, established above, that any bijection σ yielding a nonzero term in the configuration pairing $\langle \mathcal{G}\omega, \mathcal{L}v \rangle$ must give a bijection of \mathcal{A}_m subwords of ω and v . \square

Definition 4.3. \mathcal{LB} is the configuration basis of the free Lie algebra LV .

Theorem 4.2 has the following corollary, similar to the situation for Hall bases of Lie algebras.

Corollary 4.4. *Lie bracket expressions, when written in terms of the configuration basis, have integer coefficients.*

Proof. The Lie bracket expression ℓ will have $\mathcal{L}\omega$ coefficient $\langle \mathcal{G}\omega, \ell \rangle$. The configuration pairing of a graph and tree is always an integer. \square

5. EXAMPLES AND COMPUTATIONS

For simplicity we focus on the vector subspaces of LV where d basis elements are repeated n_1, \dots, n_d times. Classically, a counting argument on \mathcal{B} similar to Proposition 3.2 recovers the fine Witt formula

$$\dim \left\{ \begin{array}{l} \text{Lie brackets of } v_1, \dots, v_d \\ \text{with repetitions } n_1, \dots, n_d \end{array} \right\} = \frac{1}{n} \sum_{m|(n_1, \dots, n_d)} \mu(m) \frac{(\frac{n}{m})!}{(\frac{n_1}{m})! \cdots (\frac{n_d}{m})!}$$

where $n = \sum_i n_i$. For comparison with other Lie bases and for use in later examples, we record the configuration basis for some of these vector subspaces along with associated elements of \mathcal{B} . In the examples below, suppose V has basis $x < y < z$.

Example 5.1. The vector subspace of bracket expressions of x, y, z with each repeated twice has basis given by the following fourteen elements.

- $xyyzzz \mapsto [[[[[y, x], x], y], z], z]$.
The five words $xyzyyz$, $xyzzzy$, $xxzyyz$, $xxzyzy$, and $xxzzyy$ also have \mathcal{L} of this form.
- $xyyzzz \mapsto [[[[[y, x], z], z], [y, x]]]$
The three words $xyxzyz$, $xyxzzz$, and $xxzzyy$ also have \mathcal{L} of this form.
- $xyyzzz \mapsto [[z, x], [[y, x], y], z]$
 $\mathcal{L}xyzyxz$ is similar.
- $xyyzzz \mapsto [[[z, x], z], [[y, x], y]]$.
 $\mathcal{L}xyzzyy$ is similar.

Example 5.2. The vector subspace of bracket expressions with three x and four y has basis given by the following five elements.

- $xxxyyyy \mapsto [[[[[y, x], x], x], y], y], y]$
- $xxxyyyy \mapsto [[[[[y, x], y], y], [y, x], x]]]$
- $xxxyyyy \mapsto [[[[y, x], y], [[y, x], x], y]]]$
- $xxxyyyy \mapsto [[y, x], [[[[y, x], x], y], y]]]$
- $xxxyyyy \mapsto [[[[[y, x], y], [y, x]], [y, x]]]$

We may use the dual graph basis and the configuration pairing in order to write general Lie bracket expressions in terms of the configuration basis by applying Theorem 4.2. The following should be compared with the rewriting algorithm of [7, Ex. 4.13], which systematically applies the anti-symmetry and Jacobi identities to different parts of bracket expression in order to rewrite as a linear combination of Hall basis elements. This can take some time, since generically each application of the Jacobi identity adds one more bracket expression to which the algorithm must be applied.

Example 5.3. We will write the bracket expression $\ell = [[[x, y], [y, z]], [x, z]]$ in terms of the basis given in Example 5.1. First, note that ℓ must pair to zero with $\mathcal{G}xyyzz = x^* \leftarrow x^* \leftarrow y^* \rightarrow y^* \rightarrow z^*$, because no single cut on this graph will remove a subgraph containing only an x^* and z^* vertex. Similar reasoning shows that ℓ pairs to zero with all $\mathcal{G}\omega$ except for $\mathcal{G}xyyzzz$, $\mathcal{G}xyzyxz$, and $\mathcal{G}xzzyy$. After removing the $z^* \rightarrow x^*$ subgraph from these, we are left with $\mathcal{G}xyyz$, $\mathcal{G}xyzy$, and $\mathcal{G}xzyy$. Of these, $\mathcal{G}xzyy = x^* \leftarrow z^* \rightarrow y^* \rightarrow y^*$ must pair to zero with $[[x, y], [y, z]]$ because no single cut will remove a subgraph containing only a y^* and z^* vertex. So $[[[x, y], [y, z]], [x, z]] = k_1 \mathcal{L}xyyzzz + k_2 \mathcal{L}xyzyxz$. To find k_1 and k_2 we compute pairings $k_1 = \langle \mathcal{G}xyyzzz, [[x, y], [y, z]], [x, z] \rangle = -1$ and $k_2 = \langle \mathcal{G}xyzyxz, [[x, y], [y, z]], [x, z] \rangle = 1$.

Example 5.4. We will write the bracket expression $\ell = [[[z, y], x], x], [y, z]]$ in terms of the basis given in Example 5.1. Reasoning as before, we immediately see that ℓ pairs to zero with $\mathcal{G}xyyzzz$, $\mathcal{G}xxzzyy$, $\mathcal{G}xyyzzz$, $\mathcal{G}xzzyy$, and $\mathcal{G}xyyzzz$. For the remaining graphs, we compute pairings.

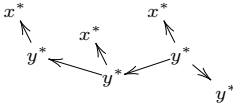
- $\langle \mathcal{G}xyzyxz, [[[z, y], x], x], [y, z] \rangle = -1$.

- $\langle \mathcal{G}xxyzzy, [[[[z, y], x], x], [y, z]] \rangle = 1.$
- $\langle \mathcal{G}xxzyyz, [[[[z, y], x], x], [y, z]] \rangle = 1.$
- $\langle \mathcal{G}xxzyzy, [[[[z, y], x], x], [y, z]] \rangle = -1.$
- $\langle \mathcal{G}xyxzyz, [[[[z, y], x], x], [y, z]] \rangle = 2.$
- $\langle \mathcal{G}xyxzzz, [[[[z, y], x], x], [y, z]] \rangle = -2.$
- $\langle \mathcal{G}xyzxzy, [[[[z, y], x], x], [y, z]] \rangle = 2.$
- $\langle \mathcal{G}xyyzxz, [[[[z, y], x], x], [y, z]] \rangle = 2.$
- $\langle \mathcal{G}xyzzyx, [[[[z, y], x], x], [y, z]] \rangle = -2.$

Pairing computations may be done either by iterating Theorem 2.8, or by applying Definition 2.3 directly, or by some combination of the two. Applying Theorem 4.2 we have the following.

$$\begin{aligned} [[[[z, y], x], x], [y, z]] &= -\mathcal{L}xyzyz + \mathcal{L}xyzzz + \mathcal{L}xxzyyz - \mathcal{L}xxzyzy \\ &\quad + 2\mathcal{L}xyxzyz - 2\mathcal{L}xyxzzz + 2\mathcal{L}xyzxzy + 2\mathcal{L}xyyzxz - 2\mathcal{L}xyzzyx \end{aligned}$$

Example 5.5. We will write the bracket expression $\ell = [[[[[x, y], y], x], x], y], y]$ in terms of the basis given in Example 5.2. Note that $\mathcal{G}xyxyxy$ must pair to zero with ℓ since we cannot make two consecutive cuts

from the graph  each time removing a single y^* vertex. For the remaining graphs,

we compute pairings.

- $\langle \mathcal{G}xyxyyy, [[[[[x, y], y], x], x], y], y] \rangle = -1.$
- $\langle \mathcal{G}xyxyyy, [[[[[x, y], y], x], x], y], y] \rangle = 1.$
- $\langle \mathcal{G}xyxyxy, [[[[[x, y], y], x], x], y], y] \rangle = 2.$
- $\langle \mathcal{G}xyxyxy, [[[[[x, y], y], x], x], y], y] \rangle = 1.$

Thus $[[[[[x, y], y], x], x], y], y] = -\mathcal{L}xyxyyy + \mathcal{L}xyxyyy + 2\mathcal{L}xyxyxy + \mathcal{L}xyxyxy.$

Note that the difficulty of these pairing calculations is more closely related to the number of repetitions than to the length of the bracket expression.

6. THE CLASSICAL LIE COALGEBRA BASIS

Classically, the free Lie algebra LV is spanned as a vector space by bracket expressions of the form $[[y_1, y_2], \dots, y_n]$. Similarly the cofree conilpotent Lie coalgebra EV^* is spanned by “long graphs” – those of the form $y_1^* \rightarrow y_2^* \rightarrow y_3^* \rightarrow \dots \rightarrow y_n^*$. Write $y_1^* | y_2^* | \dots | y_n^*$ for such a graph. From [11, Prop. 3.21], the coideal of

EV^* generated by arrow-reversing and Arnold expressions is spanned by shuffles $\sum_{\sigma \in k\text{-Shuffle}} w_{\sigma(1)} | \dots | w_{\sigma(n)}$

where k -Shuffles are the shuffles of $(1, \dots, k)$ into $(k+1, \dots, n)$. Thus, we recover a classical representation of the conilpotent Lie coalgebra implied by the equivalence of Harrison homology and commutative Andre-Quillen homology. The induced coalgebra structure, already noted in [9], is merely the anti-commutative cut coproduct. The induced configuration pairing with Lie algebras may be quickly computed by recursively applying Theorem 2.8.

In this context, the configuration pairing recovers an item of classical interest. Write $p : LV \rightarrow ULV$ for the standard map from LV to its universal enveloping algebra. Recall that the universal enveloping algebra of a free Lie algebra is canonically isomorphic to the free associative algebra TV , and in this case p is given by $p(v_i) = v_i$ on generators and $p([l_1, l_2]) = p(l_1)p(l_2) - p(l_2)p(l_1).$

Proposition 6.1. *The configuration pairing $\langle y_1^* | y_2^* | \dots | y_n^*, \ell \rangle$ is equal to the coefficient of the word $y_1 y_2 \dots y_n$ in $p(\ell).$*

Proof. This follows formally from Theorem 2.8, the structure of p noted above, and the fact that on generators $\langle v_i^*, v_j \rangle = \delta(i, j).$ \square

Write $\bar{\mathcal{G}}$ for the map $\mathcal{A} \rightarrow EV^*$ which reads a word as a bar word, $\bar{\mathcal{G}}y_1y_2\ldots y_n = y_1^*|y_2^*|\cdots|y_n^*$. Classically, Lyndon words are a multiplicative basis for the algebra of all words with shuffle product [6]. Combined with [11, Prop. 3.21] this implies $\bar{\mathcal{G}}\mathcal{B}$ is a vector space basis for EV^* .

Example 6.2. We can use the configuration pairing with $\bar{\mathcal{G}}\mathcal{B}$ to recover the result of Example 5.5. In Figure 1 we give the portion of the pairing matrix of $\bar{\mathcal{G}}\mathcal{B}$ and \mathcal{LB} for the basis recorded in Example 5.2 as well as with the Lie bracket expression from Example 5.5. To conserve space, we write $\mathcal{L}\omega_1, \dots, \mathcal{L}\omega_7$ for the (lexicographically ordered) basis of Example 5.2 and $\ell = [[[[[x, y], y], x], x], y], y]$. For visual clarity we leave blank positions where the configuration pairing is zero.

	$\mathcal{L}\omega_1$	$\mathcal{L}\omega_2$	$\mathcal{L}\omega_3$	$\mathcal{L}\omega_4$	$\mathcal{L}\omega_5$	ℓ
$\bar{\mathcal{G}}\omega_1$	-1					1
$\bar{\mathcal{G}}\omega_2$	3	1				-2
$\bar{\mathcal{G}}\omega_3$		-3	1			-1
$\bar{\mathcal{G}}\omega_4$		3	-2	1		
$\bar{\mathcal{G}}\omega_5$		6	-2	2	-1	4

$$\ell = -\mathcal{L}\omega_1 + \mathcal{L}\omega_2 + 2\mathcal{L}\omega_3 + \mathcal{L}\omega_4$$

FIGURE 1. Pairing with configuration basis from Example 5.2.

Example 6.3. In the same way, we recover the result of Examples 5.3 and 5.4. In Figure 2, we give the portion of the pairing matrix of $\bar{\mathcal{G}}\mathcal{B}$ and \mathcal{LB} for the (lexicographically ordered) basis recorded in Example 5.1. It is an exercise for the reader to finish this example by computing pairings with the brackets of Examples 5.3 and 5.4.

	$\mathcal{L}\omega_1$	$\mathcal{L}\omega_2$	$\mathcal{L}\omega_3$	$\mathcal{L}\omega_4$	$\mathcal{L}\omega_5$	$\mathcal{L}\omega_6$	$\mathcal{L}\omega_7$	$\mathcal{L}\omega_8$	$\mathcal{L}\omega_9$	$\mathcal{L}\omega_{10}$	$\mathcal{L}\omega_{11}$	$\mathcal{L}\omega_{12}$	$\mathcal{L}\omega_{13}$	$\mathcal{L}\omega_{14}$
$\bar{\mathcal{G}}\omega_1$	1													
$\bar{\mathcal{G}}\omega_2$		1												
$\bar{\mathcal{G}}\omega_3$			1											
$\bar{\mathcal{G}}\omega_4$				1										
$\bar{\mathcal{G}}\omega_5$					1									
$\bar{\mathcal{G}}\omega_6$						1								
$\bar{\mathcal{G}}\omega_7$	-2						-1							
$\bar{\mathcal{G}}\omega_8$		-2						-1						
$\bar{\mathcal{G}}\omega_9$			-2						-1					
$\bar{\mathcal{G}}\omega_{10}$							1	1	1	-1				
$\bar{\mathcal{G}}\omega_{11}$								-1	-2	2	-1			
$\bar{\mathcal{G}}\omega_{12}$								1	2			-1		
$\bar{\mathcal{G}}\omega_{13}$							-2	-1				1	-1	
$\bar{\mathcal{G}}\omega_{14}$						-2								-1

FIGURE 2. Pairing with configuration basis from Example 5.1.

The shape of the pairing matrix in the previous examples is no coincidence.

Theorem 6.4. *If $\omega < v \in \mathcal{B}$ then $\langle \bar{\mathcal{G}}\omega, \mathcal{L}v \rangle = 0$. Furthermore $\langle \bar{\mathcal{G}}\omega, \mathcal{L}\omega \rangle = \pm 1$.*

In our proof, we make use of the following computational proposition and its corollary. (To conserve space below, we neglect marking $*$ on graph vertex labels.)

Proposition 6.5. *Modulo the Arnold and arrow-reversing relations, the following local identity holds.*

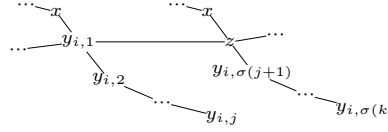
$$\begin{array}{c} \text{---} y_1 \text{---} \\ \vdots \\ \text{---} y_k \text{---} \end{array} \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} z \text{---} = (-1)^k \sum_{\sigma \in \Sigma_k} \begin{array}{c} \text{---} y_{\sigma(1)} \text{---} \\ \vdots \\ \text{---} y_{\sigma(k)} \text{---} \end{array} \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} z \text{---}$$

Corollary 6.6. *Modulo the Arnold and arrow-reversing relations, the following local identity holds.*

$$\begin{array}{c} \text{---} x \text{---} \\ \vdots \\ \text{---} x \text{---} \end{array} \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} y \text{---} = (-1)^k k! \begin{array}{c} \text{---} x \text{---} \\ \vdots \\ \text{---} x \text{---} \end{array} \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} y \text{---}$$

Proof of Theorem 6.4. Applying Theorem 4.2, the first statement is equivalent to $\bar{\mathcal{G}}\omega = \sum_i c_i \mathcal{G}v_i$ where $v_i \leq \omega$. We apply the Arnold identity repeatedly, neglecting arrows for simplicity.

Repeated applications of the Arnold identity and Corollary 6.6 convert $\cdots \text{---} y_{i,\ell_i} \text{---} x \text{---} \cdots \text{---} y_{i+1,1} \text{---} \cdots$ to a linear combination of graphs of the form $\cdots \text{---} y_{i,\ell_i} \text{---} x \text{---} \cdots \text{---} y_{i+1,1} \text{---} \cdots$ with strings of x vertices of varying lengths off of the y_{i,ℓ_i} and $y_{i+1,1}$ vertices. Similarly, applying Arnold and Proposition 6.5 converts the graph $\cdots \text{---} y_{i,1} \text{---} x \text{---} \cdots \text{---} y_{i,k} \text{---} z \text{---} \cdots$ (with either $z = y_{i,\ell_i}, k = \ell_i - 1$ or $z = y_{i+1,1}, k = \ell_i$) to a linear combination of graphs of the form



where $1 \leq j \leq k$ and σ is some permutation of $\{j+1, \dots, k\}$ (if $j = 1$ then $y_{i,1}$ has no “tail”, if $j = k$ then z has no “tail”). Note in particular that $y_{i,1} \dots y_{i,j} \leq y_{i,1} \dots y_{i,j} \dots y_{i,k}$. Applying these two steps across all of ω from left to right, combining “tails” using Proposition 6.5, and then recursing over the grading \mathcal{A}_n (letters, simple words, simple words of simple words, etc), rewrites $\bar{\mathcal{G}}\omega = \sum_i c_i \mathcal{G}v_i$ where $v_i \leq \omega$.

To check $\langle \bar{\mathcal{G}}\omega, \mathcal{L}\omega \rangle = \pm 1$, it is enough to follow through the Arnold identity applications above, keeping track of which graphs will eventually lead to $\mathcal{G}\omega$. At each step there is only a single such graph. \square

Proof of Proposition 6.5. Write G for the graph $\begin{array}{c} \text{---} y_1 \text{---} \\ \vdots \\ \text{---} y_k \text{---} \end{array} \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} z \text{---}$ and order the letters $\{y_i\}_i \cup \{z\}$ so that z is minimal.

It is enough to show $G = \sum_{\sigma \in \Sigma_k} y_{\sigma(1)} \text{---} \begin{array}{c} \nearrow \\ \rightarrow \\ \searrow \end{array} y_{\sigma(k)} \text{---} z \text{---}$ modulo Arnold and arrow-reversing, assuming z and y_i are all unique. However, this follows from the pairing calculations $\langle G, \mathcal{L}zy_{\sigma(1)} \dots y_{\sigma(k)} \rangle = 1$ and $\langle G, \mathcal{L}\omega \rangle = 0$ for all other $\omega \in \mathcal{B}$. \square

Remark 6.7. An independent proof of Theorem 6.4 could yield an alternate proof that the configuration basis is a basis without the use of graph coalgebras. However, showing 6.4 entirely in the realm of associative algebras, never using graphs, appears difficult.

Remark 6.8. Theorem 6.4 gives an independent proof that the Lyndon-Shirshov words are a basis for the Lie coalgebra EV^* . This yields an alternate proof that the Lyndon-Shirshov words are a multiplicative basis for the shuffle algebra [6].

7. COMPARISON WITH OTHER LIE BASES

In practice it is often easier to compute pairings via Theorem 2.8 using the bar basis $\bar{\mathcal{G}}\mathcal{B}$ than using the graph basis $\mathcal{G}\mathcal{B}$. The cobracket of a bar expression of length $(n + m)$ has only two terms of the form (length n expression) \otimes (length m expression) – given by cutting either after position n or after position m (and anti-commuting). On the other hand, the cobracket of a graph expression could have many such terms from cutting various edges. However, moving to the bar representation of Lie coalgebras gives up the monomial dual basis $\mathcal{G}\mathcal{B}$.

Example 7.1. Write $[\omega]$ for the classical bracketing method constructing a Hall basis from Lyndon words. Recall from [7] that this is recursively defined by $[\omega] = [[\alpha], [\beta]]$ where $\omega = \alpha\beta$ is chosen so that α is nonempty and β is lexicographically minimal (classically it is equivalent to choose β so that it is the longest possible such Lyndon subword). For example, $[xxxxyyy] = [x, [x, [[[x, y], y], y], y]]$ and $[xyyyxyy] = [[x, [[x, y], y]], [[x, y], y]]$. Using this basis, Example 6.2 is as in Figure 3. For further comparison, in Figure 4 we include also the pairing matrix corresponding to Example 6.3.

	$[\omega_1]$	$[\omega_2]$	$[\omega_3]$	$[\omega_4]$	$[\omega_5]$	ℓ
$\bar{\mathcal{G}}\omega_1$	1					1
$\bar{\mathcal{G}}\omega_2$	-4	1				-2
$\bar{\mathcal{G}}\omega_3$	6	-3	1			-1
$\bar{\mathcal{G}}\omega_4$	-4	2	-2	1		
$\bar{\mathcal{G}}\omega_5$		3	-2	3	1	4

$$\ell = [\omega_1] + 2[\omega_2] - [\omega_3] - 2[\omega_4] + 2[\omega_5]$$

FIGURE 3. Pairing with Lyndon basis for words of Example 5.2.

	$[\omega_1]$	$[\omega_2]$	$[\omega_3]$	$[\omega_4]$	$[\omega_5]$	$[\omega_6]$	$[\omega_7]$	$[\omega_8]$	$[\omega_9]$	$[\omega_{10}]$	$[\omega_{11}]$	$[\omega_{12}]$	$[\omega_{13}]$	$[\omega_{14}]$
$\bar{\mathcal{G}}\omega_1$	1													
$\bar{\mathcal{G}}\omega_2$	-2	1												
$\bar{\mathcal{G}}\omega_3$		-1	1											
$\bar{\mathcal{G}}\omega_4$		-1		1										
$\bar{\mathcal{G}}\omega_5$	2	1	-2	-2	1									
$\bar{\mathcal{G}}\omega_6$	-1		1	1	-1	1								
$\bar{\mathcal{G}}\omega_7$			-1				1							
$\bar{\mathcal{G}}\omega_8$			2		-1		-2	1						
$\bar{\mathcal{G}}\omega_9$			-1		1	-2	1	-1	1					
$\bar{\mathcal{G}}\omega_{10}$						1			-1	1				
$\bar{\mathcal{G}}\omega_{11}$				-1	1	-2		-1	2	-2	1			
$\bar{\mathcal{G}}\omega_{12}$		2			-2	4		1	-2			1		
$\bar{\mathcal{G}}\omega_{13}$				2	-1			1			-2	-1	1	
$\bar{\mathcal{G}}\omega_{14}$				-1	1	-2					-1		1	1

FIGURE 4. Pairing with Lyndon basis for words of Example 5.1.

It immediately follows from [7, Thm. 5.1] and Proposition 6.1 that the configuration pairing of $\bar{\mathcal{G}}\mathcal{B}$ and $[\mathcal{B}]$ is always lower triangular with 1 on the diagonal, similar to Theorem 6.4 for the configuration basis. Other bases, such as the right-normed basis $[T_X]$ of [1] do not satisfy a triangularity theorem such as Theorem 6.4. This is verified via explicit pairing calculations.

Example 7.2. The basis $[\mathcal{B}]$ of [1] satisfies a triangular pairing theorem [1, Prop. 4.1]. Since [1] uses Shirshov's ordering convention for Lyndon-Shirshov words, we reverse the ordering of the alphabet \mathcal{A}_0 for comparison with other bases. Chibrikov's basis is very similar to the configuration basis – when the leading letter is repeated few times, many of these basis elements will differ by only a sign, for example $\mathcal{L}xyzxyz = [[z, x], [[y, x], z], y]$ and $[[xyz]yz] = [[[[x, y], z], y], [x, z]]$. For comparison, we give its analogous pairing matrices below. (Below, we use the reverse ordering on the generators of V to account for the use of Shirshov's definition of \mathcal{B} in [1].)

	$[\omega_1]$	$[\omega_2]$	$[\omega_3]$	$[\omega_4]$	$[\omega_5]$	$[\omega_6]$	$[\omega_7]$	$[\omega_8]$	$[\omega_9]$	$[\omega_{10}]$	$[\omega_{11}]$	$[\omega_{12}]$	$[\omega_{13}]$	$[\omega_{14}]$
$\bar{\mathcal{G}}\omega_1$	1													
$\bar{\mathcal{G}}\omega_2$		1												
$\bar{\mathcal{G}}\omega_3$			1											
$\bar{\mathcal{G}}\omega_4$				1										
$\bar{\mathcal{G}}\omega_5$					1									
$\bar{\mathcal{G}}\omega_6$						1								
$\bar{\mathcal{G}}\omega_7$	-2						1							
$\bar{\mathcal{G}}\omega_8$		-2						1						
$\bar{\mathcal{G}}\omega_9$			-2						1					
$\bar{\mathcal{G}}\omega_{10}$							-1			1				
$\bar{\mathcal{G}}\omega_{11}$										-1	1			
$\bar{\mathcal{G}}\omega_{12}$												1		
$\bar{\mathcal{G}}\omega_{13}$							2						1	
$\bar{\mathcal{G}}\omega_{14}$						-2								1

FIGURE 5. Pairing with the Lyndon basis for words of Example 5.1.

Note that the Lyndon basis $\bar{\mathcal{G}}\mathcal{B}$ does not in general have a dual monomial basis of bracket expressions. This can be verified by computing the pairings of the Lyndon basis with all bracket expressions in the vector subspace where x is repeated twice and y is repeated three times.

	$[[[[x, y], x], y], y]$	$[[[[x, y], y], x], y]$	$[[[[x, y], y], y], x]$	$[[[x, y], y], [x, y]]$
$\mathcal{G}xxyyy$	-1	-1	-1	
$\bar{\mathcal{G}}xyxyy$	2	2	3	-1

FIGURE 6. Lyndon words have no dual basis of monomials.

8. A NEW SHUFFLE BASIS

Work similar to Theorem 4.2 and 6.4 can be used to construct other bases of associative words for EV^* similar to the Lyndon-Shirshov words. This also yields new multiplicative bases for the shuffle algebra.

We use the ordering on words called *degree-lexicographic* or *deg-lex* by [1]. In this ordering, $\omega < v$ if and only if either ω has less letters than v , or else ω and v have the same number of letters and $\omega < v$ lexicographically. For a finite alphabet, this is equivalent to using the ordering of letters to view words as numbers; e.g. in the alphabet $\{1 < 2\}$ we have $2 < 12 < 21 < 112$.

Recall (see Example 3.4) that words have unique expression as a word of compatible simple words. Write \prec for the ordering of \mathcal{A} given by the *deg-lex* ordering of words in the *deg-lex* ordered alphabet of simple words (thus $(13322) \prec (13)(122) \prec (122)(13)$ unlike when working lexicographically). It is clear that \prec is a total order on \mathcal{A} .

Definition 8.1. Let $\hat{\mathcal{B}}$ be the set of finite words which have minimal \prec ordering among their cyclic permutations. Define $\hat{\mathcal{B}}_m = \hat{\mathcal{B}} \cap \mathcal{A}_m$ and $\mathcal{L}\hat{\mathcal{B}}$, $\mathcal{G}\hat{\mathcal{B}}$ the same as in Section 4.

The set $\hat{\mathcal{B}}_m$ satisfies Proposition 3.2 and Lemma 3.5 which is enough to make the proof of Theorem 4.2 apply.

Theorem 8.2. *Bracket expressions $\mathcal{L}\hat{\mathcal{B}}$ and graph expressions $\mathcal{G}\hat{\mathcal{B}}$ are dual vector space bases for LV and EV^* .*

Furthermore, Lie bracket expressions, when written in terms of $\mathcal{L}\hat{\mathcal{B}}$, have integer coefficients.

In the proof of Theorem 6.4, applications of the Arnold identity and Proposition 6.5 and Corollary 6.6 reduce the \prec ordering as well as the lexicographical ordering, so an analog of Theorem 6.4 holds for $\hat{\mathcal{B}}$.

Theorem 8.3. *If $\omega \prec v \in \hat{\mathcal{B}}$ then $\langle \bar{\mathcal{G}}\omega, \mathcal{L}v \rangle = 0$. Furthermore $\langle \bar{\mathcal{G}}\omega, \mathcal{L}\omega \rangle = \pm 1$.*

Corollary 8.4. *The expressions $\bar{\mathcal{G}}\hat{\mathcal{B}}$ are a vector space basis of EV^* .*

Corollary 8.5. *The words of $\hat{\mathcal{B}}$ are a multiplicative basis for the shuffle algebra.*

Example 8.6. In an alphabet with only two letters, \mathcal{B} and $\hat{\mathcal{B}}$ are the same.

In the vector subspace of brackets x , y , and z each repeated twice, the change from \mathcal{B} to $\hat{\mathcal{B}}$ affects the following words. $\omega_{11} = xyzyxz$ is replaced by $\hat{\omega}_{10} = xzxyyz$ and $\omega_{13} = xyzzyx$ is replaced by $\hat{\omega}_{11} = xzxyzy$ and $\omega_{10} \cdots \omega_{14}$ are reordered.

The corresponding portion of the pairing matrix for $\bar{\mathcal{G}}\hat{\mathcal{B}}$ and $\mathcal{L}\hat{\mathcal{B}}$ has the same number of non-zero off-diagonal entries as that of $\bar{\mathcal{G}}\mathcal{B}$ and $[\mathcal{B}]$.

	$\mathcal{L}\hat{\omega}_1$	$\mathcal{L}\hat{\omega}_2$	$\mathcal{L}\hat{\omega}_3$	$\mathcal{L}\hat{\omega}_4$	$\mathcal{L}\hat{\omega}_5$	$\mathcal{L}\hat{\omega}_6$	$\mathcal{L}\hat{\omega}_7$	$\mathcal{L}\hat{\omega}_8$	$\mathcal{L}\hat{\omega}_9$	$\mathcal{L}\hat{\omega}_{10}$	$\mathcal{L}\hat{\omega}_{11}$	$\mathcal{L}\hat{\omega}_{12}$	$\mathcal{L}\hat{\omega}_{13}$	$\mathcal{L}\hat{\omega}_{14}$
$\bar{\mathcal{G}}\hat{\omega}_1$	1													
$\bar{\mathcal{G}}\hat{\omega}_2$		1												
$\bar{\mathcal{G}}\hat{\omega}_3$			1											
$\bar{\mathcal{G}}\hat{\omega}_4$				1										
$\bar{\mathcal{G}}\hat{\omega}_5$					1									
$\bar{\mathcal{G}}\hat{\omega}_6$						1								
$\bar{\mathcal{G}}\hat{\omega}_7$	-2						-1							
$\bar{\mathcal{G}}\hat{\omega}_8$		-2						-1						
$\bar{\mathcal{G}}\hat{\omega}_9$			-2						-1					
$\bar{\mathcal{G}}\hat{\omega}_{10}$				-2						-1				
$\bar{\mathcal{G}}\hat{\omega}_{11}$					-2						-1			
$\bar{\mathcal{G}}\hat{\omega}_{12}$						-2						-1		
$\bar{\mathcal{G}}\hat{\omega}_{13}$							1	1	1				-1	
$\bar{\mathcal{G}}\hat{\omega}_{14}$								1	2					-1

FIGURE 7. Pairing analogous to Example 5.1.

It is not possible to extend Chibrikov's definition of $[\![\omega]\!]$ to $\hat{\mathcal{B}}$; however the classical bracketing $[\omega]$ does extend to $\hat{\mathcal{B}}$. In fact, even the proof of the triangularity theorem [7, Thm. 4.9] appears to extend to $[\hat{\mathcal{B}}]$. The result of pairing $\bar{\mathcal{G}}\hat{\mathcal{B}}$ and $[\hat{\mathcal{B}}]$ is similar to Figure 4.

The basis $\mathcal{L}\hat{\mathcal{B}}$ has a good property with respect to vector space quotient maps. Let $\phi : V \rightarrow W$ be a vector space quotient map, and suppose that V and W have ordered bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ compatible with ϕ so that $\phi : \{v_1, \dots, v_n\} \rightarrow \{w_1, \dots, w_m\}$ with $\phi(v_i) \leq \phi(v_j)$ for $i < j$. Suppose further that $\phi^{-1}(w_1) = \{v_1\}$. Write $\hat{\mathcal{B}}_V$, $\hat{\mathcal{B}}_W$ for the sets of words $\hat{\mathcal{B}}$ with respect to the ordered alphabets $\{v_i\}$

and $\{w_j\}$. Define the map $\hat{\phi} : \hat{\mathcal{B}}_V \rightarrow \hat{\mathcal{B}}_W \cup \{0\}$ by $\hat{\phi}(a_1 \cdots a_k) = \phi(a_1) \cdots \phi(a_k)$ if $\phi(a_1) \cdots \phi(a_k) \in \hat{\mathcal{B}}_W$ and 0 otherwise.

Proposition 8.7. *The map $L\phi : LV \rightarrow LW$ is given on the basis \mathcal{LB}_V by $\mathcal{L}\hat{\phi} : \mathcal{LB}_V \rightarrow \mathcal{LB}_W \cup \{0\}$*

The above proposition can be used to quickly calculate \mathcal{LB} . For example, $[[[v_2, v_1], v_2], [v_2, v_1]] \in \mathcal{LB}$ so $[[[v_i, v_1], v_j], [v_k, v_1]] \in \mathcal{LB}$ as well for all $i, j, k \neq 1$. Neither $[\mathcal{B}]$, $[[\mathcal{B}]]$, nor $[\hat{\mathcal{B}}]$ have this property.

9. FUTURE DIRECTIONS

9.1. Pairing matrix formula. The off-diagonal elements of the pairing matrix of $\bar{\mathcal{G}}\mathcal{B}$ and the configuration basis \mathcal{LB} are due to applications of the Arnold identity in the proof of Theorem 6.4. A more careful analysis, keeping track of signs and counting occurrences should lead to a explicit formulas writing $\bar{\mathcal{G}}\mathcal{B}$ in terms of $\mathcal{G}\mathcal{B}$, which could be used to write the pairing matrix without any pairing computations. This would yield marked computational improvements, since there are no known formulas giving the analogous pairing matrix for either $[\mathcal{B}]$ or $[[\mathcal{B}]]$. Since the applications of Arnold in Theorem 6.4 run from left to right, this computation will be simplest for $\hat{\mathcal{B}}$ which gathers small simple words on the left side.

9.2. Gröbner basis implementation. The triangularity theorems (6.4 and 8.3) imply that \mathcal{LB} and \mathcal{LB} may be used in Gröbner basis calculations. This should be implemented in a computer algebra software platform such as *Sage* or *GAP*.

9.3. Dual monomial basis. It is unclear whether there is no dual monomial basis of Lie bracket expressions of LV and associative (bar) words of EV^* . As noted in Figure 6, the Lyndon-Shirshov words do not have a dual monomial basis. Neither will purely left-normed bracket expressions (e.g. those of the form $[[[\cdot, \cdot], \cdot], \cdot]$ – expressions with 3 of one generator and 4 of another provides a counter-example). However, ad-hoc dual monomial bases can be found for examples of computable size.

APPENDIX A. BASES

Below we list the bases used in Figures 2, 4, and 5 presented previously. Recall that in our computation of $[[\mathcal{B}]]$, we reversed the order of the basis elements to account for the use of Shirshov's ordering convention by [1].

	$\mathcal{L}\omega$	$[\omega]$	$[[\omega]]$
$\omega_1 = xxyyzz$	$[[[[[y, x], x], y], z], z]$	$[x, [x, [y, [y, z], z]]]$	$[[[[[x, [x, y]], y], z], z]$
$\omega_2 = xxyzzy$	$[[[[[y, x], x], z], y], z]$	$[x, [[x, [y, z]], [y, z]]]$	$[[[[[x, [x, y]], z], y], z]$
$\omega_3 = xxyzzy$	$[[[[[y, x], x], z], z], y]$	$[x, [[x, [y, z], z]], y]$	$[[[[[x, [x, y]], z], z], y]$
$\omega_4 = xxzyyz$	$[[[[[z, x], x], y], y], z]$	$[x, [[x, z], [y, [y, z]]]]$	$[[[[[x, [x, z]], y], y], z]$
$\omega_5 = xxzyyz$	$[[[[[z, x], x], y], z], y]$	$[x, [[[x, z], [y, z]], y]]$	$[[[[[x, [x, z]], y], z], y]$
$\omega_6 = xxzzyy$	$[[[[[z, x], x], z], y], y]$	$[x, [[[x, z], z], y], y]$	$[[[[[x, [x, z]], z], y], y]$
$\omega_7 = xyxyzx$	$[[[[y, x], z], z], [y, x]]$	$[x, y], [x, [[y, z], z]]$	$[[[x, y], [[x, y], z]], z]$
$\omega_8 = xyxzyz$	$[[[[z, x], y], z], [y, x]]$	$[x, y], [[x, z], [y, z]]$	$[[[[x, y], [x, z]], y], z]$
$\omega_9 = xyxzyy$	$[[[[z, x], z], y], [y, x]]$	$[x, y], [[[x, z], z], y]$	$[[[[x, y], [x, z]], z], y]$
$\omega_{10} = xyxzzz$	$[[[z, x], z], [[y, x], y]]$	$[[[x, y], y], [[x, z], z]]$	$[[[[x, y], y], [x, z]], z]$
$\omega_{11} = xyxzzx$	$[z, x], [[y, x], y], z]$	$[x, [y, [y, z]], [x, z]]$	$[[[x, y], y], z], [x, z]]$
$\omega_{12} = xyzxzy$	$[[z, x], y], [[y, x], z]$	$[x, [y, z]], [[x, z], y]$	$[[[[x, y], z], [x, z]], y]$
$\omega_{13} = xyzxzy$	$[z, x], [[y, x], z], y]$	$[[[x, [y, z]], y], [x, z]]$	$[[[[x, y], z], y], [x, z]]$
$\omega_{14} = xzxzyy$	$[[[z, x], y], y], [z, x]]$	$[x, z], [[[x, z], y], y]$	$[[[x, z], [[x, z], y]], y]$

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