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2013 – 09 – 26

Tropical Geometry I

→ Amoebas of algebraic varieties

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $(\mathbb{C}^*)^n \simeq$ n dim'l complex algebraic torus

Say $V \subset (\mathbb{C}^*)^n$ is a subvariety

$$\underset{\mathbb{R}^n}{\log(V)} = \{ (\log|z_1|, \dots, \log|z_n|) \text{ where } (z_1, \dots, z_n) \in V \}$$

This is the "<u>amoeba of V</u>"

<u>Prop:</u> $\log(V)$ is closed in $(\mathbb{C}^*)^n$.

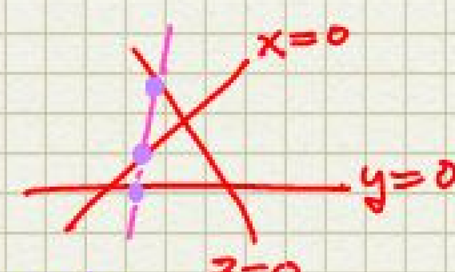
↳ "Think of it like a manifold w/ boundary"

$V \subset (\mathbb{C}^*)^n \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$

what is $\bar{V}$ in here?

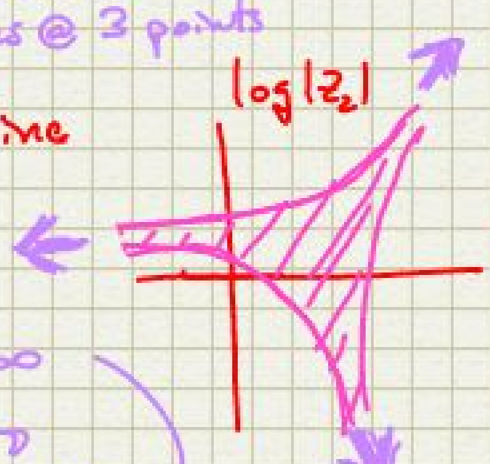
↳ Intersects coordinate hyperplanes of $\mathbb{P}^n(\mathbb{C})$

<u>EX</u> $\mathbb{CP}^2 = \mathbb{P}^2(\mathbb{C})$

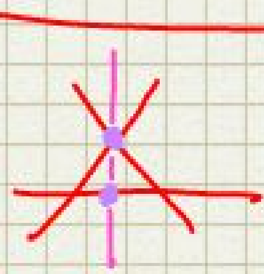


(Line intersects coord planes @ 3 points)

Amoeba of line

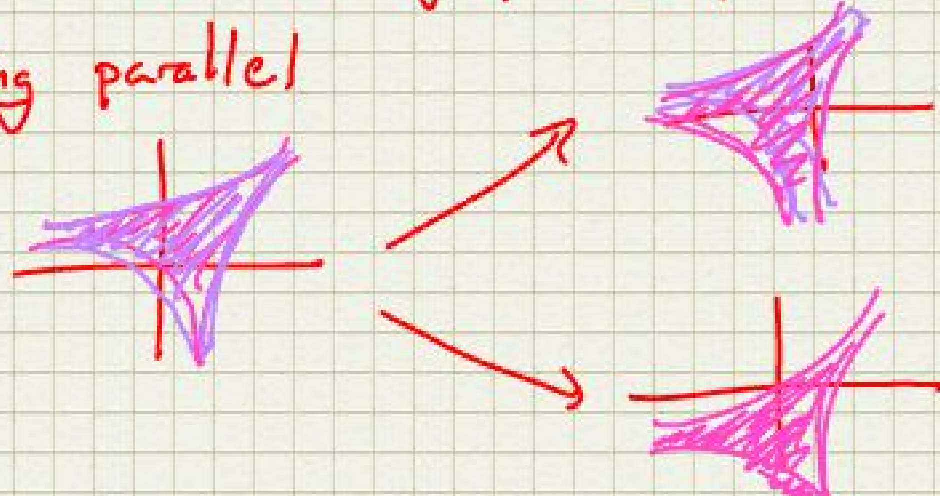


(going out to ∞ corresponds to intersection points)



⇝ Degenerate amoeba

→ When the line changes, the position changes by shifting parallel



(and maybe squeezing or stretching)

<u>Note</u>: Amoeba is not intrinsic to V → it depends also on <u>embedding</u> $V \subset (\mathbb{C}^*)^n$

- Very little is known about amoebas of varieties of codim ≥ 2
- $\dim(\log V) = \dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V$ $\Big\{$ for a generic variety V

<u>but</u> if $n \leq 2\dim_{\mathbb{C}} V$, then

$$\dim(\log V) = n$$ ← "not enough space"

Amoebas of hypersurfaces

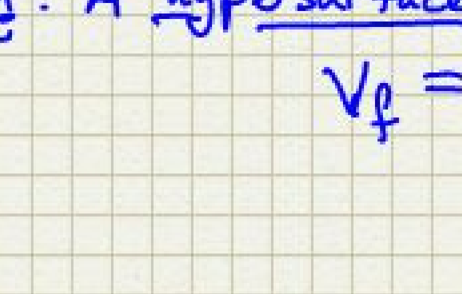
<u>Recall</u>: A <u>hypersurface</u> is

$$V_f = \{f = 0\} \subset (\mathbb{C}^*)^n$$

$$\hookrightarrow f \in \mathbb{C}[z_i^{\pm 1}]_{i=1}^n$$

<u>Thm</u>: Each connected component of $\mathbb{R}^n \setminus \log(V_f)$ is convex.

<u>EX</u>



<u>Def</u>: <u>Newton polytope</u> of $f = \sum_{\bar{m}} c_{\bar{m}} z^{\bar{m}}$ is the convex hull of $\bar{m}$ w/ $c_{\bar{m}} \neq 0$

<u>EX</u> $f(x,y) = 1 + x^2 + x^2y^2 + y^2$ ⇝

Thm: # connected components of $\mathbb{R}^n \setminus V_f$

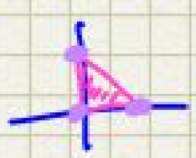
$\leq$

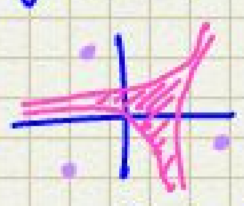
# of lattice points in Newton polytope

$$\psi : \begin{array}{ccc} \text{connected components} & \longrightarrow & \text{lattice points} \\ \cup & & \cup \\ \text{bounded components} & \longmapsto & \text{interior points} \end{array}$$

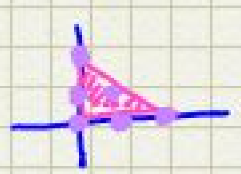
Plane curves!

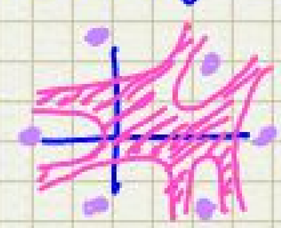
deg 1. line $ax+by+c=0$



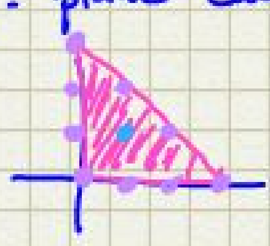


deg 2. plane conic $ax^2+by^2+\cdots=0$





deg 3. plane cubics $ax^3+by^3+\cdots=0$



Nonsingular deg d curves

$$g = \tfrac{1}{2}(d-1)(d-2)$$
$$= \#\left(\text{interior points of Newton polytope}\right)$$

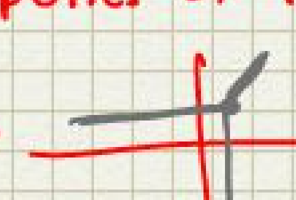
Cor: $\#(\text{interior holes}) \leq g$

Problem: Amoebas are too difficult to compute

↳ Can we get some simpler (i.e. piece-wise linear) objects using amoebas which reflect essential properties of V.

Spines

$\rightsquigarrow$



$\boxed{\text{"Tropical Line"}}$

2013.10.03 (Tropical Geometry Lecture 2)

Different approaches to tropical varieties.

→ **Tropical Semifield**

$\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ with operations $x \oplus y = \max\{x, y\}$
$x \odot y = x + y$

"semi-field"

- $\mathbb{T}$ is a commutative semigroup under $\oplus$ w/ $id_{\oplus} = -\infty$
  - → too many idempotents to be made into a group ($x \oplus x = x$ for all x)
- $\odot$ is commutative, associative, distributes over $\oplus$; multiplicative inverses exist for $x \in \mathbb{T} \setminus \{-\infty\}$

Define new operations: $x \oplus_t y = \log_t(t^x + t^y)$ for $t > 0$

$x \odot_t y = \underbrace{x + y}_{= \log_t(t^x \cdot t^y)}$ $\quad x, y \in \mathbb{R}$

(This is the $\log_t$ image of ordinary addition & multiplication on $\mathbb{R}_{\geq 0}$)

Connection to previous operations:
$$\left[\begin{array}{l} \lim_{t \to \infty} x \odot_t y = x \odot y \\ \lim_{t \to \infty} x \oplus_t y = \max\{x, y\} = x \oplus y \end{array}\right]$$
"Maslov dequantization"

→ **Tropical Polynomials.**

$A \subset (\mathbb{Z}_{\geq 0})^n$ ← multi-index

$f(x) = \bigoplus_{j \in A} a_j \odot x^{\odot j}$ ← circled things are "tropical operations"

$= \max_{j \in A} (a_j + \langle j, x \rangle)$ $\quad a_j \in \mathbb{T}$

EX: $f(x) = a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2}$
$= \max\{a_0, a_1 + x, a_2 + 2x\}$

Note: Graph is always going to be a convex region.

Zero Locus.

Classical theory: $V_f = \{x \text{ where } f(x) = 0\}$

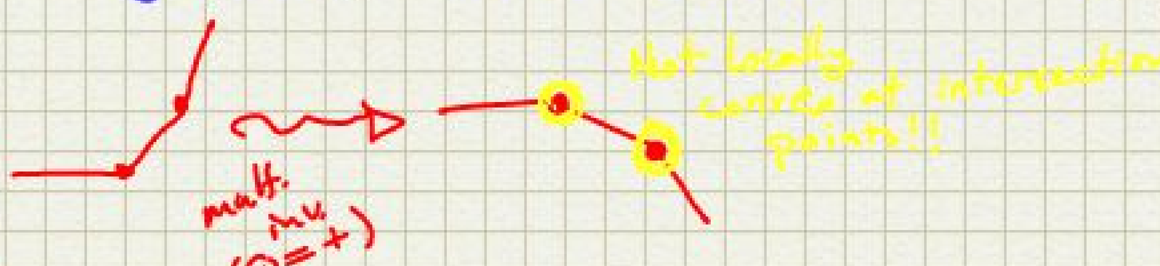
Tropical: Attempt #1 $V_f = \{x \text{ where } f(x) = -\infty\}$ ← $\oplus$ is max, so you won't see much.

Alternative.

Classical theory: $V_f = \{x \text{ where } 1/f \text{ is not regular}\}$

Tropical: $V_f = \left\{ \begin{array}{l} x \text{ where mult. inverse of } f \text{ is } \underline{\underline{\text{not}}} \\ \text{locally convex at } x \end{array} \right\}$

Ex



$\Longrightarrow V_f = \underline{\text{corner locus}}$ of graph of f !!

Ex $a_0 \oplus a_1 \odot x \oplus a_2 \odot x^{\odot 2}$

Ex: $f(x,y) = 2 \odot x \oplus 3 \odot y \oplus 5$
$= \max\{2+x,\ 3+y,\ 5\}$

Families of Amoebas

Let V_t be a family of varieties depending on t

$$V_t = \{z \in (\mathbb{C}^*)^n \text{ where } \sum_{\substack{j \in A \\ A \subset \mathbb{Z}^n_{\geq 0}}} a_j(t) z^j = 0\}$$

(Remark: Members of this family may be non-isomorphic.)

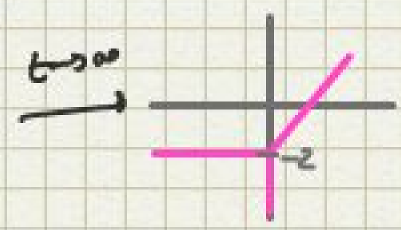
Consider $\mathrm{Log}_t : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$ by

$(z_1, \dots, z_n) \longmapsto (\log_t |z_1|, \dots, \log_t |z_n|)$ ← This is amoeba map scaled by $1/\ln t$

Ex: V_t = constant family; eg line

problem: This also goes to same thing!

Ex $V_t = \{x + t^2 y = 1\}$



We want to look at limit $\lim_{t \to \infty}(\mathrm{Log}_t(V_t))$

$\rightsquigarrow$ Use Hausdorff metric.

Given A, B sets in metric space w/metric d

define $d_H(A,B) = \max(d(A,B), d(B,A))$

where $d(A,B) = \sup_{a \in A} d(a,B)$

& $d(a,B) = \inf_{b \in B} d(a,b)$

"Amount you must thicken A or B to contain the other"

We say that $\{A_t\}$ converges to A if for all compact $K \subset X$ there is open $U \supset K$ so that

$$\lim_{t \to \infty}\left(d_H(A_t \cap U, A \cap U)\right) = 0$$

Thm: For any family V_t of varieties, (where coeff's depend algebraically on t) $\mathrm{Log}_t(V_t)$ is convergent.

Def: Thing it converges to is a "tropical variety"

Kapranov's Thm: For the case of hypersurfaces, these two notions of "tropical variety" are the same.

→ Non-Archimedian amoebas

Instead of thinking of $V_t \subset (\mathbb{C}^*)^n$ as a family depending on t, change the base field:

$$K = \bigcup_n \overline{\mathbb{C}(t^{1/n})}$$

Look at $V \subset K^n$

$\mathrm{Log}_t \longleftrightarrow$ Valuation

$\lim_{t \to \infty}$

$v: K \to \mathbb{R}$

$K^n \to \mathbb{R}^n$

Image of V of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$: "Non-Archimedian Amoeba"

↳ get back tropical varieties